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A STUDY OF FACTORS AFFECTING THE ACCURACY OF POSITION FIX FOR LUNAR TRAJECTORIES

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SUMMARY

The accuracy of the position fix as obtained from simultaneous onboard optical measurements is discussed with respect to the gradients of the measurements. Equations for the gradients of various types of measurements in position fixing are discussed. An approximate method for predicting the error ellipsoid for a position-fixing system is given. The position errors are illustrated by application of the results to a lunar trajectory.

INTRODUCTION

As part of a study in circumlunar navigation, equations were developed in reference 1 for fixing a position from various combinations of simultaneous onboard optical measurements. For several of these combinations, the resultant errors in position fix due to the addition of random errors in measurement were investigated by means of a Monte Carlo technique, and the results were reported in reference 2. The purpose of the present paper is to extend the results of references 1 and 2 by establishing some simple criteria for selecting those measurements which give the best estimate of position, and by developing a simplified procedure for estimating the accuracy of the position fix.

The ability to fix the position within prescribed limits is determined by the accuracy of the measurements, which in turn is dependent upon the accuracy of the instruments and the rates of change of the measurements. The rate of change of the measurement with position, referred to as the gradient of the measurement, has direction as well as magnitude. From a knowledge of the gradients, the direction of maximum and minimum accuracy for a given combination of measurements can be determined. The direction of maximum accuracy (direction of minimum error) and the magnitude of the error should be considered in any position-fixing system. For example, if it is desired to establish an accurate altitude when a vehicle is orbiting close to the earth, moon, or some other celestial body, a position-fixing system should have its direction of maximum accuracy along the radius vector from the vehicle to the body center; that is, the system should be sensitive to changes in this radial distance. For earth-moon trajectories where it is desirable to stay in a corridor, the direction of maximum accuracy of the system should be normal to the corridor so that any straying from the corridor may be remedied. When accurate velocity determination is desired from position data, the direction of maximum accuracy of the position-fixing system should be along the velocity vector.

The linear relation between position errors and measurement errors for simultaneous onboard optical measurements permits the use of small-perturbation theory in error analysis. (See ref. 2.) Hence, the present paper is based on this theory. The equations relating measurement deviation and position deviation from the nominal for simultaneous sightings are stated. The combined effect of measurement error and the gradient of the measurement upon position error is also presented. Standard linear error analysis techniques are used to determine the error distribution, that is, the error ellipsoid within which lies a given percentage of the probable errors resulting from a position-fixing system. Subsequently, an error parallelepiped is shown as a simplified method of approximating the error distribution. The position errors are illustrated by application of the results to a lunar trajectory.

SYMBOLS

$\left[A\right]$	orthogonal rotation matrix of eigenvectors of $\left[\sum \epsilon_r\right]$
$\bar{h}$	gradient of measurement
$\left[I\right]$	identity matrix
$\bar{i},\bar{j},\bar{k}$	unit vectors along X-, Y-, and Z-axes
K^2	values of chi square for a given probability that the error lies inside the error ellipsoid
m	measurement
Δm	measurement deviation from nominal, $\frac{\partial m}{\partial x}(\Delta x) + \frac{\partial m}{\partial y}(\Delta y) + \frac{\partial m}{\partial z}(\Delta z)$
ϵm	measurement error
R	radius of body on which measurements are being made
r	distance from vehicle to body center
$\bar{r}_s$	unit vector in star direction
$\epsilon\bar{r}$	position-error vector

t	time
Δt	clock error or error in knowledge of time
X,Y,Z	vehicle-centered right-hand rectangular coordinate axes oriented in same direction as earth's axes (i.e., with X-axis in direction of Aries and Z-axis in direction of celestial north pole)
x,y,z	coordinates of a point in X,Y,Z system
$\epsilon x, \epsilon y, \epsilon z$	error in measurement of x, y, and z
X',Y',Z'	vehicle-centered normal axes of error ellipsoid
x',y',z'	coordinates of a point in X',Y',Z' system
α	one-half of angular diameter of body as viewed from vehicle
β	angle between gradient of a measurement and position-error vector
γ	angle between radius vector to center of a body and gradient of angle included at vehicle between a star and nearest horizon point of body
θ	angle included at vehicle between two stars
λ	eigenvalue of covariance matrix of position errors
$\left[\sum \epsilon_m\right]$	covariance matrix of measurement errors
$\left[\sum \epsilon_r\right]$	covariance matrix of position errors in X,Y,Z system
$\left[\sum \epsilon_{r'}\right]$	covariance matrix of position errors in X',Y',Z' system
σ	standard deviation, or standard error
$\sigma_{x'}, \sigma_{y'}, \sigma_{z'}$	standard error in measurement of x', y', and z'

Subscripts:

1,2,3	refer to measurements 1, 2, and 3
s	star direction

Notations:

—	indicates a vector

$\cdot$	indicates a scalar multiplication
$\times$	indicates a vector multiplication
$\lvert\ \rvert$	absolute value
$[\ ]$	square matrix
$[\ ]^{-1}$	inverse of square matrix
$[\ ]^{t}$	transpose of square matrix
$\{\ \}$	column matrix
$\left.\frac{\partial m}{\partial x}\right\rvert_{x}$	partial derivative of measurement with respect to x evaluated at x
$\left.\frac{\partial m}{\partial x}\right\rvert_{x+\delta x}$	partial derivative of measurement with respect to x evaluated at $x + \delta x$

VEHICLE POSITION DETERMINATION

Measurement Gradients

Because of fuel and time schedule requirements, circumlunar vehicles will be required to fly very close to a nominal trajectory such as that of reference 3 shown in figure 1. The vehicle position in an earth-centered system is given by the following equation:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{vehicle} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}_{nominal} + \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} \tag{1}$$

The vehicle position in the general case is determined from optical and time measurements made onboard. Under these conditions the vehicle optical and time measurement deviations from the nominal trajectory will vary linearly with the vehicle position deviations from the nominal trajectory as shown by:

$$\Delta m = \frac{\partial m}{\partial x}(\Delta x) + \frac{\partial m}{\partial y}(\Delta y) + \frac{\partial m}{\partial z}(\Delta z) + \frac{\partial m}{\partial t}(\Delta t) \tag{2}$$

Four measurements are required to solve for Δx, Δy, Δz, and Δt in equation (2). However, for the circumlunar trip it can be assumed that the clock error Δt is zero and that the measurement deviations reduce effectively to functions of x, y, and z. Thus, three simultaneous optical measurements at a given time t may be used to determine the vehicle deviations from the nominal trajectory. The equation used to determine the vehicle deviation is

$$\begin{Bmatrix} \Delta m_1 \\ \Delta m_2 \\ \Delta m_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial m_1}{\partial x} & \frac{\partial m_1}{\partial y} & \frac{\partial m_1}{\partial z} \\ \frac{\partial m_2}{\partial x} & \frac{\partial m_2}{\partial y} & \frac{\partial m_2}{\partial z} \\ \frac{\partial m_3}{\partial x} & \frac{\partial m_3}{\partial y} & \frac{\partial m_3}{\partial z} \end{bmatrix} \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix} \tag{3}$$

where the gradient of a measurement is

$$\text{grad } m = \bar{h} = \bar{i}\,\frac{\partial m}{\partial x} + \bar{j}\,\frac{\partial m}{\partial y} + \bar{k}\,\frac{\partial m}{\partial z} \tag{4}$$

Schematic drawings of the gradient directions of possible optical measurements in a vehicle-centered coordinate system are shown in figure 2. Although these measurements are essentially of two general types, that is, angular diameter and star-to-body, several examples are shown as illustrations. Equations for the gradients are developed in the appendix. In the present paper the term declination denotes the declination of a body (positive north from the XY-plane of the vehicle), and the term right ascension denotes the right ascension of a body as measured in the XY-plane of the vehicle (positive east from the X-axis).

Error Index

A navigation measurement may be broken down into three parts: (a) the value which should be measured if the vehicle were on the nominal trajectory, (b) the increment of measurement due to the increment of vehicle position from the nominal trajectory, and (c) the actual error in measurement. The relation between measurement error and position error is essentially linear over a wide range of measurement error (ref. 2) and may be written as:

$$\epsilon m = \frac{\partial m}{\partial x}(\epsilon x) + \frac{\partial m}{\partial y}(\epsilon y) + \frac{\partial m}{\partial z}(\epsilon z) \tag{5}$$

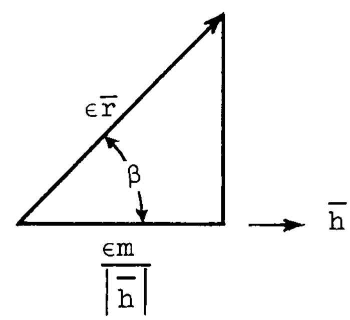
The derivatives in this equation may be evaluated at the point x,y,z since, for small increments, $\left.\frac{\partial m}{\partial x}\right|_x = \left.\frac{\partial m}{\partial x}\right|_{(x+\delta x)}$, and so forth. Hence, all errors in the present paper were calculated as though the vehicle were on the nominal trajectory at the time of measurement.

Equation (5) may also be written:

$$\epsilon m = |\bar{h}||\epsilon\bar{r}|\cos\beta \tag{6}$$

where $\epsilon\bar{r}$ is the position error, $\bar{h}$ is the gradient of the measurement, and β is the angle between $\bar{h}$ and $\epsilon\bar{r}$. Thus, the projection of the position-error vector $\epsilon\bar{r}$ on the gradient vector is equal to $\frac{\epsilon m}{|\bar{h}|}$ as illustrated in the sketch:

The contribution of a measurement error to an error in position is a vector of magnitude $\frac{\epsilon m}{|\bar{h}|}$ having the direction of the gradient of the measurement. It follows that the ratio $\frac{\epsilon m}{|\bar{h}|}$ may be regarded as an error index of the measurement.



If three measurements are used in fixing the position, the position error is given by the general equation

$$\begin{Bmatrix} \epsilon x \\ \epsilon y \\ \epsilon z \end{Bmatrix} = \begin{bmatrix} \frac{\partial m_1/\partial x}{|h_1|} & \frac{\partial m_1/\partial y}{|h_1|} & \frac{\partial m_1/\partial z}{|h_1|} \\ \frac{\partial m_2/\partial x}{|h_2|} & \frac{\partial m_2/\partial y}{|h_2|} & \frac{\partial m_2/\partial z}{|h_2|} \\ \frac{\partial m_3/\partial x}{|h_3|} & \frac{\partial m_3/\partial y}{|h_3|} & \frac{\partial m_3/\partial z}{|h_3|} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\epsilon m_1}{|h_1|} \\ \frac{\epsilon m_2}{|h_2|} \\ \frac{\epsilon m_3}{|h_3|} \end{Bmatrix} \tag{7}$$

For orthogonal gradients, the equation for the magnitude of the position error reduces to:

$$\epsilon r = \sqrt{\left(\frac{\epsilon m_1}{|h_1|}\right)^2 + \left(\frac{\epsilon m_2}{|h_2|}\right)^2 + \left(\frac{\epsilon m_3}{|h_3|}\right)^2} \tag{8}$$

Thus, it can be seen that for fixed measurement errors and fixed gradient directions, the magnitude of the position errors is governed by the magnitude of the gradients of the measurements. In general, navigation systems employing measurements with the largest gradients will be the most accurate.

Except for the case of horizon measurements, the instrument error, that is, standard deviation of the instrument or some integral multiple of it, may be considered to be the measurement error ϵm. However, when measurements are made to the horizon of a body, the indefiniteness of the horizon contributes an additional measurement error to the basic instrument error so that

$$\sigma_m = \sqrt{\sigma_i^2 + \frac{\sigma_R^2}{r^2 - R^2}} \tag{9}$$

where σ_m is the standard deviation of the measurement, σ_i is the instrument error, and σ_R is a constant which varies with the radius of the body. In the present paper, σ_i is considered to be 5×10^{-5} radian and σ_R, 3.219 kilometers for the earth and 0.805 kilometer for the moon (see ref. 4), except for one case where the standard deviation is 150 arc seconds for all measurements.

Error Ellipsoid

With the assumption that the errors of position-fixing systems in circumlunar space are linear with respect to the measuring error for the range of measurement errors expected (ref. 2), standard linear error analysis techniques may be used to establish the error distribution, that is, the error ellipsoid associated with any position-fixing system (refs. 5 and 6).

The direction of maximum accuracy (i.e., minimum error) of the measurement system lies along the minor axis of the error ellipsoid, and the direction of minimum accuracy, along the major axis. The error ellipsoid may be adjusted to contain any given percentage of all probable position errors. The procedure for obtaining it is described briefly in the following paragraphs.

For convenience, the error distribution is considered in terms of zero means and is centered at the assumed vehicle center on the nominal trajectory. The covariance matrix of position errors $\left[\sum\epsilon_r\right]$ is related to the covariance matrix of measurement errors $\left[\sum\epsilon_m\right]$ as follows:

$$\left[\sum\epsilon_r\right] = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}^{-1} \left[\sum\epsilon_m\right] \left[\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}^{-1}\right]^t \tag{10}$$

where $\left[\sum\epsilon_m\right]$ is known from the characteristics of the instruments.

The theory of matrices permits an orthogonal transformation to a new set of coordinates in which the new position errors are linearly independent. The procedure is to expand the determinant of $\left|\left[\sum\epsilon_r\right] - \left\lceil I \right\rfloor \left\{\lambda\right\}\right|$ which yields a cubic in λ, the roots of which are the eigenvalues of $\left[\sum\epsilon_r\right]$. (See ref. 7.) Let the normalized eigenvectors associated with these eigenvalues form the rows of the matrix $\left[A\right]$ which is an orthogonal rotation matrix $\left(\text{because } \left[\sum\epsilon_r\right] \text{ is symmetric}\right)$. It is to be noted that the matrix $\left[\sum\epsilon_r\right]$ is proportional to the reciprocal of the matrices of references 5 and 6. Thus, the eigenvectors, $\left[A\right]$, are equal to the eigenvectors of the reference matrices whereas the eigenvalues, $\left\{\lambda\right\}$, are the reciprocals of the eigenvalues of the reference matrices. (See ref. 7.)

If a new X',Y',Z' coordinate system is established by

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \left[A\right] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

then the covariance matrix of position errors in the X',Y',Z' system is given by

$$\left[\sum\epsilon_{r'}\right] = \left[A\right]\left[\sum\epsilon_r\right]\left[A\right]^t \tag{11}$$

and $\left[\sum\epsilon_{r'}\right]$ is a diagonal matrix of $\sigma_{x'}^2$, $\sigma_{y'}^2$, and $\sigma_{z'}^2$.

The frequency function of the errors is given by:

$$f(x',y',z') = \frac{1}{(2\pi)^{3/2}\sigma_{x'}\sigma_{y'}\sigma_{z'}} e^{-\frac{1}{2}\left(\frac{x'^2}{\sigma_{x'}^2}+\frac{y'^2}{\sigma_{y'}^2}+\frac{z'^2}{\sigma_{z'}^2}\right)} \tag{12}$$

The exponent $\left(\frac{x'^2}{\sigma_{x'}^2} + \frac{y'^2}{\sigma_{y'}^2} + \frac{z'^2}{\sigma_{z'}^2}\right)$ has a chi-square distribution with three degrees of freedom. The equation of the error ellipsoid in the rotational reference X',Y',Z' system is given by:

$$\frac{x'^2}{\sigma_{x'}^2} + \frac{y'^2}{\sigma_{y'}^2} + \frac{z'^2}{\sigma_{z'}^2} = K^2 \tag{13}$$

The values of K^2 for various probabilities that the position error lies within the ellipsoid may be found from probability tables for a chi-square distribution. (See table I.)

The error ellipsoid is given in the X,Y,Z system by:

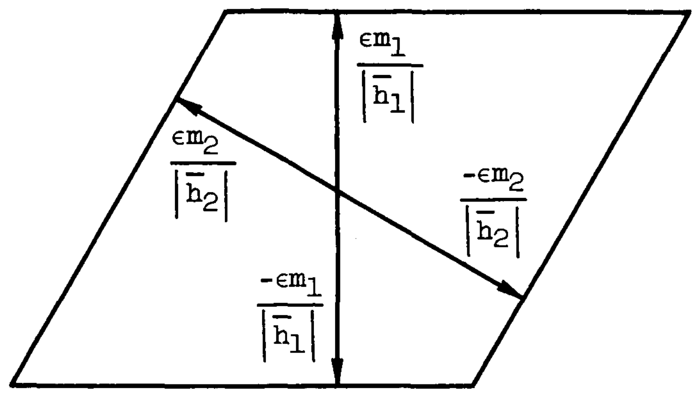
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} A \end{bmatrix}^{-1} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} A \end{bmatrix}^{t} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \tag{14}$$

The concept of the error ellipsoid is applicable to any number of dimensions. (See ref. 5.) Thus, if clock error is present, the equation for the error ellipsoid may be revised to include this. The value of chi square will then be chosen from tables for a problem with four degrees of freedom.

Error Parallelepiped

As indicated in the previous section, the calculation of the error ellipsoid is rather complex and lengthy. A simpler procedure for estimating the error ellipsoid and its orientation is obtained from consideration of the measurement error and the gradients of the measurement field. As shown in the section entitled "Error Index," the projection of the position-error vector $\epsilon\bar{r}$ on the gradient of the measurement $\bar{h}$ gives the error index of the measurement.

Thus, for two measurements the following error geometry is obtained:



The quadrilateral contains all position errors due to measurement errors equal to or less than $\pm\epsilon m_1$ and $\pm\epsilon m_2$ with gradients $\bar{h}_1$ and $\bar{h}_2$, respectively. If three measurements are involved, the figure becomes a parallelepiped, but the same procedures apply.

Thus, if the gradient direction and error index of each measurement are known, then an error parallelepiped may be constructed by passing planes normal to each gradient at a distance from the origin equal to the magnitude of the error index. If measurement errors of 1σ are assumed, the error parallelepiped contains all 1σ position errors due to all three measurements.

RESULTS AND DISCUSSION

Error Ellipsoid and Error Parallelepiped

Figure 3 presents a 3σ error ellipsoid and a 3σ error parallelepiped calculated from measurements of angular diameter, declination, and right ascension. These were calculated for a vehicle on the outbound leg of the lunar trajectory shown in figure 1. For these calculations, the earth's coordinates were: $x = 16{,}100$ kilometers, $y = 104{,}610$ kilometers, and $z = 59{,}550$ kilometers. A standard error of 150 arc seconds was assumed for all measurements.

Also shown in figure 3(a) are 200 position errors obtained by the Monte Carlo method of error analysis. It is of interest to note that all except four of the position errors fall within the error parallelepiped. These four position errors are also outside the error ellipsoid boundary.

A comparison of the error ellipsoid and error parallelepiped shows that the direction of the gradients of the measurements defines the orientation of the axes of both the error ellipsoid and the error parallelepiped.

The maximum position error calculated by means of an error parallelepiped and an error ellipsoid is shown in figure 4. The position error was calculated from measurements of the angular diameter of the earth and the angles included at the vehicle between each of two stars and the earth's center. A 1σ instrument error was assumed. The vehicle was outbound on the lunar trajectory shown in figure 1. Figure 4 shows that the position error calculated by means of the error ellipsoid is considerably greater than that calculated by means of the error parallelepiped. If a 3σ instrument error had been assumed in the calculation of position error, the curves would be in much better agreement, as can be shown by equations (7) and (13).

Correction for Nonsimultaneous Measurements

The navigation error has been calculated by using small-perturbation theory with the assumption that measurements are made simultaneously. However, when navigation measurements are made manually, simultaneous measurements by a single operator are impracticable. Hence, the increment in time must be taken into account in determining the position fix (ref. 1) as well as the error distribution.

Figure 5 shows rates of change of various measurements with time, calculated at points along the lunar trajectory shown in figure 1. The curves shown in figure 5 were obtained from the following equation:

$$\frac{dm}{dt} = \frac{\partial m}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial m}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial m}{\partial z}\frac{\partial z}{\partial t} \tag{15}$$

In the figure a solid line indicates that dm/dt is positive; a broken line, that dm/dt is negative; and a skip, that dm/dt changes sign. From the figure it can be seen that dm/dt varies considerably with distance as well as with type of measurement.

Over a large part of the trajectory this type of figure may be used to provide a correction to the measurements used in a position-fix system when the increment of time between measurements is small. For example, if the declination and angular diameter of earth are measured at time t_d and t_a, respectively, and it is desired to find the position and the error distribution at time t_a, the declination may be corrected by taking the declination read at time t_d and adding to it a correction $(t_a - t_d)$ multiplied by the average rate of change of the declination for the time interval chosen.

The position-fix equations are then solved with the angular measurements corrected to time t_a. The error distribution is calculated by using the values of x, y, and z obtained from the corrected equations. Over a large portion of the trajectory, if the time between measurements is of the order of 6 minutes or less, this method gives a good first-order correction to compensate for the effect of nonsimultaneous measurements.

Selection of Measurements

The direction of maximum accuracy of the measurement system has been shown to lie along the minor axis of the error ellipsoid or the minimum axis of the error parallelepiped. This direction can be determined from a knowledge of the error indices and of the gradient directions of the measurements used.

The magnitudes of the gradients of several measurements which may be made of the earth or the moon from the vehicle during a lunar trip are shown in figure 6. The measurements are: angular diameter, declination and right ascension of the earth or moon as seen from the vehicle, and the angles included at the vehicle between the lines of sight from the vehicle (a) to a star and to the earth (or moon) center, (b) to a star and to the nearest horizon point of earth (or moon), and (c) to a star and a beacon orbiting about the earth (or moon). The use of star-and-landmark techniques would result in gradients equivalent to those obtained by using star-to-body-center techniques. The gradients shown in figure 6 were calculated at points along the trajectory shown in figure 1. Equations for both magnitude and direction of the gradients are given in the appendix.

Figure 7 shows the error indices of some of the measurements whose gradients are shown schematically in figure 2. The measurement error for an assumed instrument error of 1σ and the gradients of the measurements were calculated at various points on the outbound portion of the trajectory shown in figure 1. Both measurement errors and gradients vary widely along the trajectory with the greater variation occurring in the gradients. It is therefore clear that measurements with large gradients should be used if the error is to be kept a minimum. Even though instrument error is very small, the position error will be large if the gradients of the measurements are also very small.

Figure 7 shows that the type of measurements yielding the greatest accuracy may change rapidly. From the figure it is seen that there is a point on the trajectory where moon measurements result in greater accuracy than earth measurements, that is, $\left|\frac{\epsilon m}{\bar{h}}\right|_{\text{moon}} < \left|\frac{\epsilon m}{\bar{h}}\right|_{\text{earth}}$. This point occurs somewhere between 50 and 55 percent of the distance from earth to moon for all measurements investigated except the angular diameter. The error indices of the angular diameter (too large to be shown in the figure) indicate that the switch from earth to moon measurements should take place about 69 percent of the distance. Since these results pertain to single measurements, it should be noted that the point at which moon measurements yield the greater accuracy will vary when a combination of measurements is used. (See ref. 2.)

In any position-fixing system, those measurements whose gradients have the proper direction and magnitude to assure the desired direction of maximum accuracy should be made. In cases where it is impossible to obtain a measurement gradient in the desired direction with a single measurement, two measurements may be combined (not necessarily a proportional combination) to form a single measurement. The resultant gradient is the vector sum of the gradients of the measurements included in the combination and is governed by the proportionality factors. These measurements are subject to the same error analysis as before; however, in the present paper no analysis has been made of the resultant error of such combinations.

Position Error

By using different methods of fixing a position, two attempts were made to find the effect of the gradient directions relative to each other upon position error. The results are shown in figures 8 and 9. In figure 8 the errors were calculated by using gradients of the angular diameter and of the angles included at the vehicle between each of two stars and the earth's center. Both stars were in the XZ-plane, and the earth was on the Y-axis. One star was rotated through the angle θ about the line of sight from the vehicle to the other star. Thus, the gradient of the angular-diameter measurement was always perpendicular to the gradients of the angles between each of the two stars and the earth's center. The gradients of the angles between each of the two stars and the earth's center were separated by the angle θ. (See appendix.) It is to be noted that for this case (for error-ellipsoid calculations) the only factor affecting the error is r, the radial distance of the vehicle from the earth's center. The error indices for both the angular-diameter and star-to-body-center measurements are functions of r, but as shown in figure 7

$$\left|\frac{\epsilon m}{\bar{h}}\right|_{\text{angular diameter}} \gg \left|\frac{\epsilon m}{\bar{h}}\right|_{\text{star-to-body-center}}$$

Therefore, the angular-diameter measurement contributes practically all the error to the position vector, and the effect of the relative directions of the gradients of the other two measurements is indiscernible.

Figure 9 shows the effect upon the position error of gradients approaching a coplanar condition. The error was calculated from the gradients of the angles included at the vehicle between the earth's horizon and each of three stars equally spaced about the vehicle in the XZ-plane. The earth was located on the Y-axis, and measurements were made at various points between the earth and moon. The angle γ, defined as the angle between the gradient of the measurement and the radius vector to the earth's center, is equal for all three measurements and varies along the trajectory. Since the error index $\left|\frac{\epsilon m}{\bar{h}}\right|$ also varies with distance, the error has been divided by the error index in order to remove the effect of random error and gradient magnitude from the results. The ratio of error to error index is seen to be a maximum when the gradients are coplanar ($\gamma = 0$). The minimum value occurs at $\gamma = 54.7^{\circ}$ where the gradients of the three measurements are orthogonal.

CONCLUDING REMARKS

On a lunar trajectory the accuracy of position fix is dependent upon the instrument accuracy and the particular measurements used. The types of measurements as well as the particular combinations of measurements giving the greatest accuracy vary considerably along the trajectory. The construction of an error parallelepiped provides a simple means for estimating the accuracy of a position fix. The error parallelepiped which has the same orientation as the error ellipsoid is easily determined from a knowledge of gradient magnitude and direction.

In general, the following criteria should be observed in order that the error may be a minimum for a position-fixing system based on simultaneous onboard optical measurements in earth-moon space:

(a) The minor axis of the error ellipsoid should be oriented to the desired direction of greater accuracy in position fix.

(b) The error indices should be small. This implies large gradients for the measurements. It follows that since gradients of angular diameter are small over a large portion of a lunar trajectory, other measurements should be used over most of the trajectory, if possible.

(c) If possible, gradients of measurements used should be orthogonal. In no case should they be coplanar.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 16, 1963.

APPENDIX

GRADIENT EQUATIONS

The gradients of the measurements control the error of position. The gradients are determined by taking the directional derivatives of the measurements. For instance, when the vehicle center is taken as the origin of the coordinates, the gradient of α is given by

$$\bar{h} = \bar{i}\,\frac{\partial\alpha}{\partial x} + \bar{j}\,\frac{\partial\alpha}{\partial y} + \bar{k}\,\frac{\partial\alpha}{\partial z} \tag{A1}$$

Substituting

$$\alpha = \tan^{-1}\frac{R}{\sqrt{r^2 - R^2}} \tag{A2}$$

into equation (A1) gives the gradient of α as

$$\bar{h} = \bar{i}\left[\frac{-Rx}{r^2\left(r^2 - R^2\right)^{1/2}}\right] + \bar{j}\left[\frac{-Ry}{r^2\left(r^2 - R^2\right)^{1/2}}\right] + \bar{k}\left[\frac{-Rz}{r^2\left(r^2 - R^2\right)^{1/2}}\right] \tag{A3}$$

and the magnitude of the gradient as

$$\left|\bar{h}\right| = \frac{R}{r\left(r^2 - R^2\right)^{1/2}} \tag{A4}$$

where R is the radius of the body on which measurements are being made. From the equation $\left|\bar{h}\right|\left|\bar{r}\right|\cos\beta = \bar{h}\cdot\bar{r}$ it can be shown that the gradient lies along the radius vector connecting the vehicle with the center of the body. This gradient vector is shown diagrammatically in figure 2(a).

The gradient vector for the angle included at the vehicle between a star and the body center (fig. 2(b)) is given by

$$\bar{h} = \bar{i}\left[\frac{x\left(\bar{r} \cdot \bar{r}_s\right) - x_s r^2}{X}\right] + \bar{j}\left[\frac{y\left(\bar{r} \cdot \bar{r}_s\right) - y_s r^2}{X}\right] + \bar{k}\left[\frac{z\left(\bar{r} \cdot \bar{r}_s\right) - z_s r^2}{X}\right] \tag{A5}$$

where $\bar{r}_s$ refers to a unit vector in the star direction

$$X = r^2\sqrt{r^2 - \left(\bar{r} \cdot \bar{r}_s\right)^2} \tag{A6}$$

hence, the magnitude of the gradient is

$$\left|\bar{h}\right| = \frac{1}{r} \tag{A7}$$

Since

$$\bar{h} \cdot \bar{r} = 0 \tag{A8}$$

and

$$\bar{h} \cdot \left(\bar{r} \times \bar{r}_s\right) = 0 \tag{A9}$$

it follows that the gradient of the measurement is in the plane of the radius vector and the star and is perpendicular to the radius vector.

The gradient vector for declination (fig. 2(c)) is given by

$$\bar{h} = \bar{i}\left[\frac{-zx}{r^2\left(r^2 - z^2\right)^{1/2}}\right] + \bar{j}\left[\frac{-zy}{r^2\left(r^2 - z^2\right)^{1/2}}\right] + \bar{k}\left[\frac{r^2 - z^2}{r^2\left(r^2 - z^2\right)^{1/2}}\right] \tag{A10}$$

and the magnitude of the gradient is

$$\left|\bar{h}\right| = \frac{1}{r} \tag{A11}$$

This gradient is in the plane of the radius vector and the Z-axis and is perpendicular to the radius vector.

The gradient vector for right ascension (fig. 2(d)) is given by

$$\bar{h} = \bar{i}\left(\frac{-y}{x^2 + y^2}\right) + \bar{j}\left(\frac{x}{x^2 + y^2}\right) + \bar{k}(0) \tag{A12}$$

and the magnitude of the gradient is

$$\left|\bar{h}\right| = \frac{1}{\left(x^2 + y^2\right)^{1/2}} \tag{A13}$$

It can be shown that this gradient is parallel to the earth's equatorial plane and perpendicular to the radius vector.

For star-to-horizon measurements (fig. 2(e)), the gradient vector is given by

$$\bar{h} = \bar{i}\left[\frac{x\left(\bar{r} \cdot \bar{r}_s\right) - x_s r^2}{X} + \frac{Rx}{r^2\left(r^2 - R^2\right)^{1/2}}\right] + \bar{j}\left[\frac{y\left(\bar{r} \cdot \bar{r}_s\right) - y_s r^2}{X} + \frac{Ry}{r^2\left(r^2 - R^2\right)^{1/2}}\right]$$
$$+ \bar{k}\left[\frac{z\left(\bar{r} \cdot \bar{r}_s\right) - z_s r^2}{X} + \frac{Rz}{r^2\left(r^2 - R^2\right)^{1/2}}\right] \tag{A14}$$

where X has been previously defined. Thus,

$$|h| = \frac{1}{\left(r^2 - R^2\right)^{1/2}} \tag{A15}$$

This gradient is perpendicular to the line connecting the vehicle with the horizon of the body and lies in the plane of the vehicle, star, and horizon point nearest the star.

For a star to a given point x,y,z, such as an orbiting beacon or landmark (fig. 2(f)), the gradient of the angle included at the vehicle between a star and a point may be obtained from the formula for the gradient of the angle included at the vehicle between a star and body center by substituting the coordinates of the point for the coordinates of the body center. The gradient vector is in the plane of the star, the vehicle, and the point, and is perpendicular to the radius vector drawn from the vehicle to the point.

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TABLE I

VALUES OF K^2 FOR VARIOUS PROBABILITIES THAT

THE POSITION ERROR LIES WITHIN THE ERROR ELLIPSOID

Probability, percent	K^2 (a)
25	1.212
50	2.365
68	3.534
75	4.109
90	6.250
95	7.812
99	11.343

[a]Values of K^2 may be found in a chi-square distribution table for three degrees of freedom.

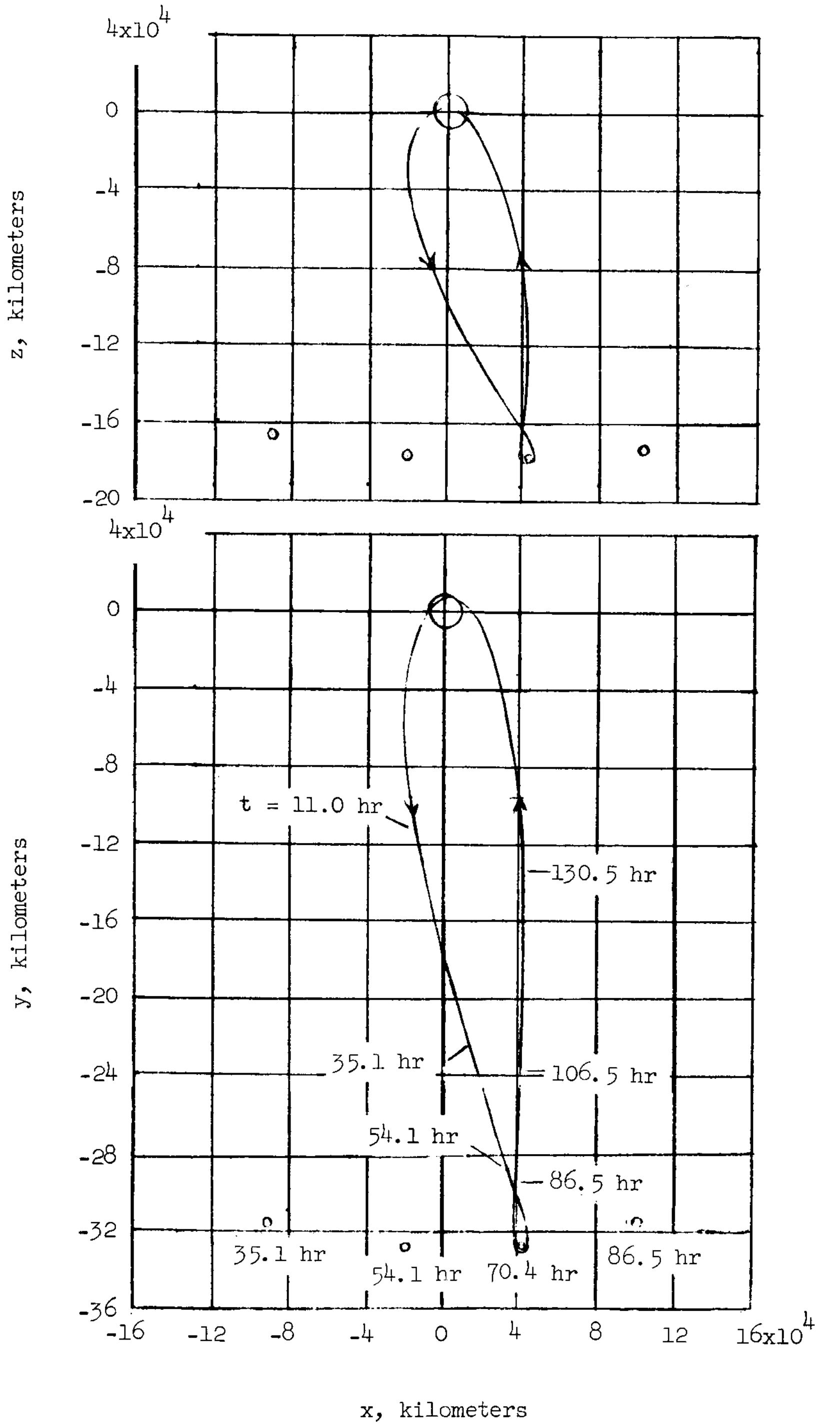


Figure 1.- Lunar trajectory used in analysis of navigation methods. Trajectory 2 of reference 3.

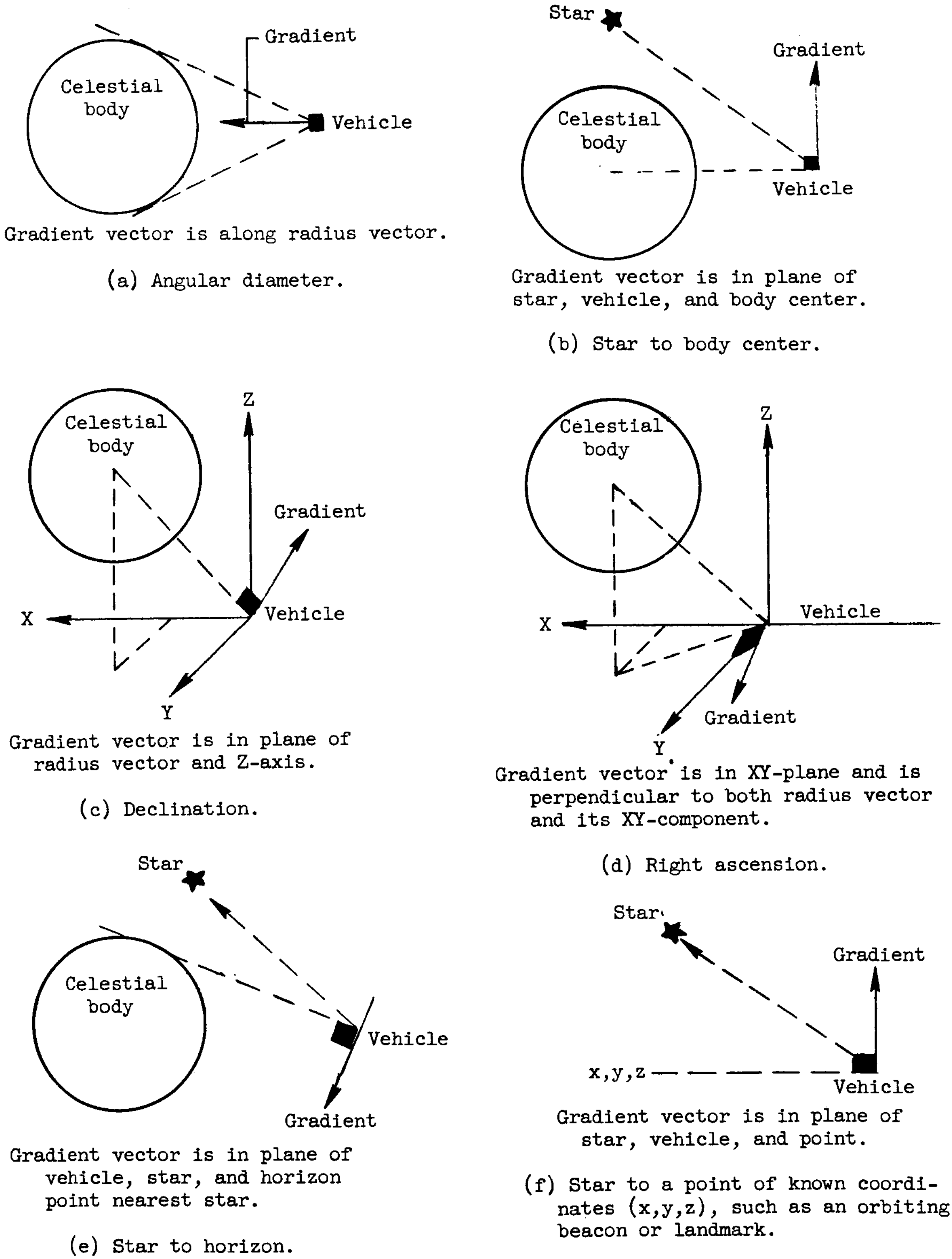
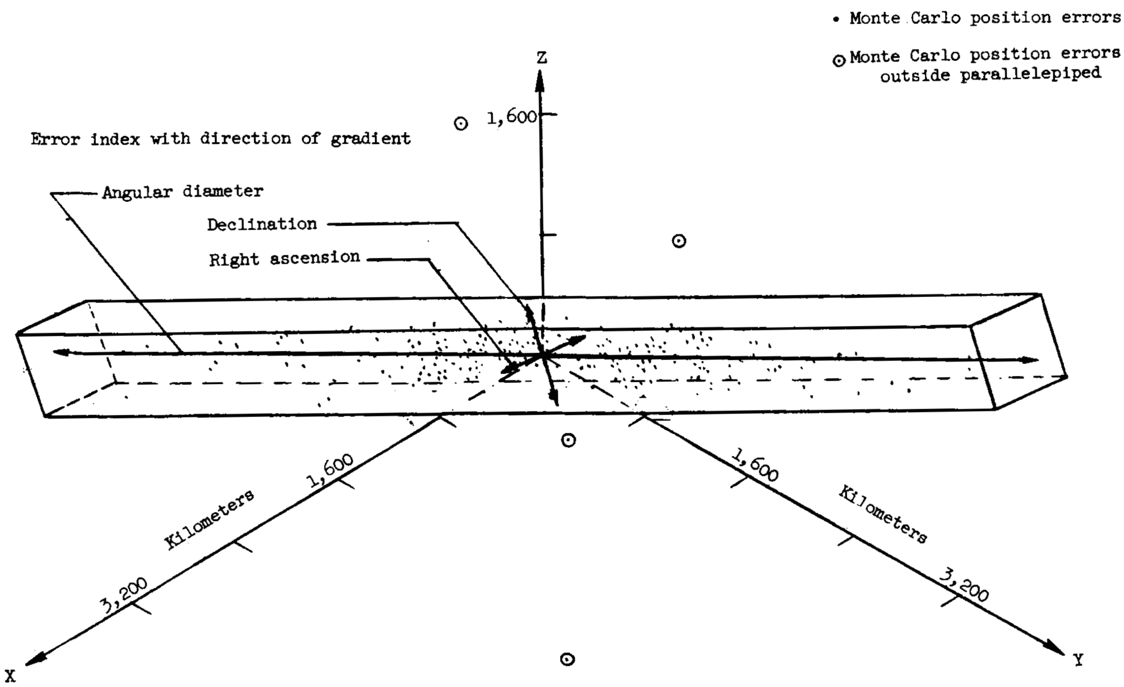
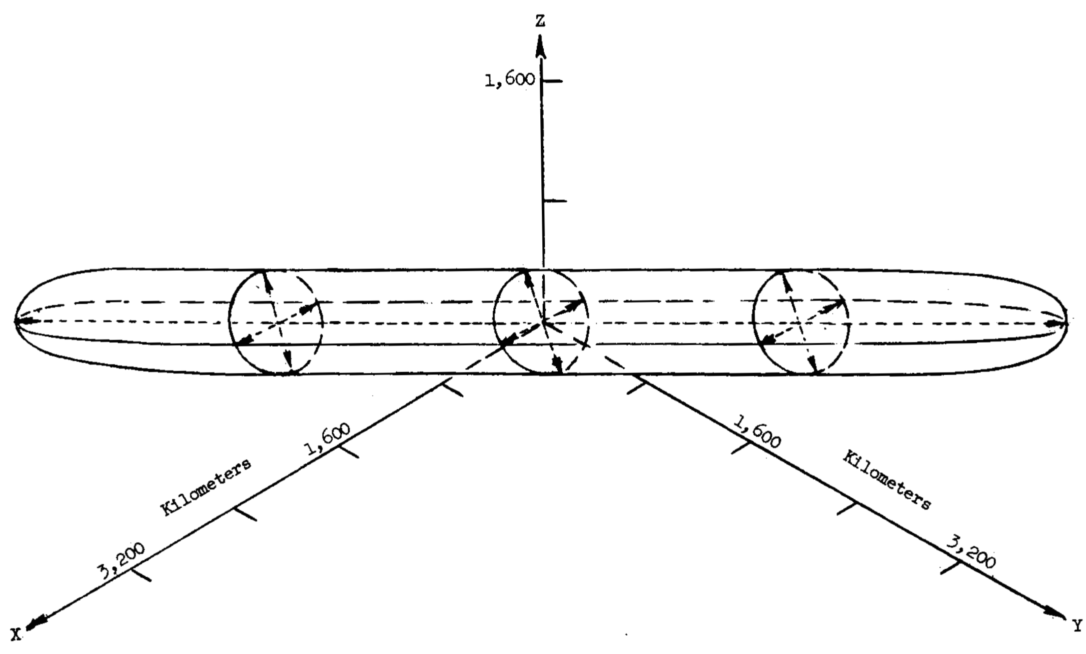


Figure 2.- Geometry of gradients of various angular measurements.



(a) 3σ error parallelepiped and 200 Monte Carlo position errors.



(b) 3σ error ellipsoid.

Figure 3.- 3σ error ellipsoid and 3σ error parallelepiped calculated from measurements of angular diameter, declination, and right ascension, assuming $\sigma = 150$ arc seconds for all measurements.

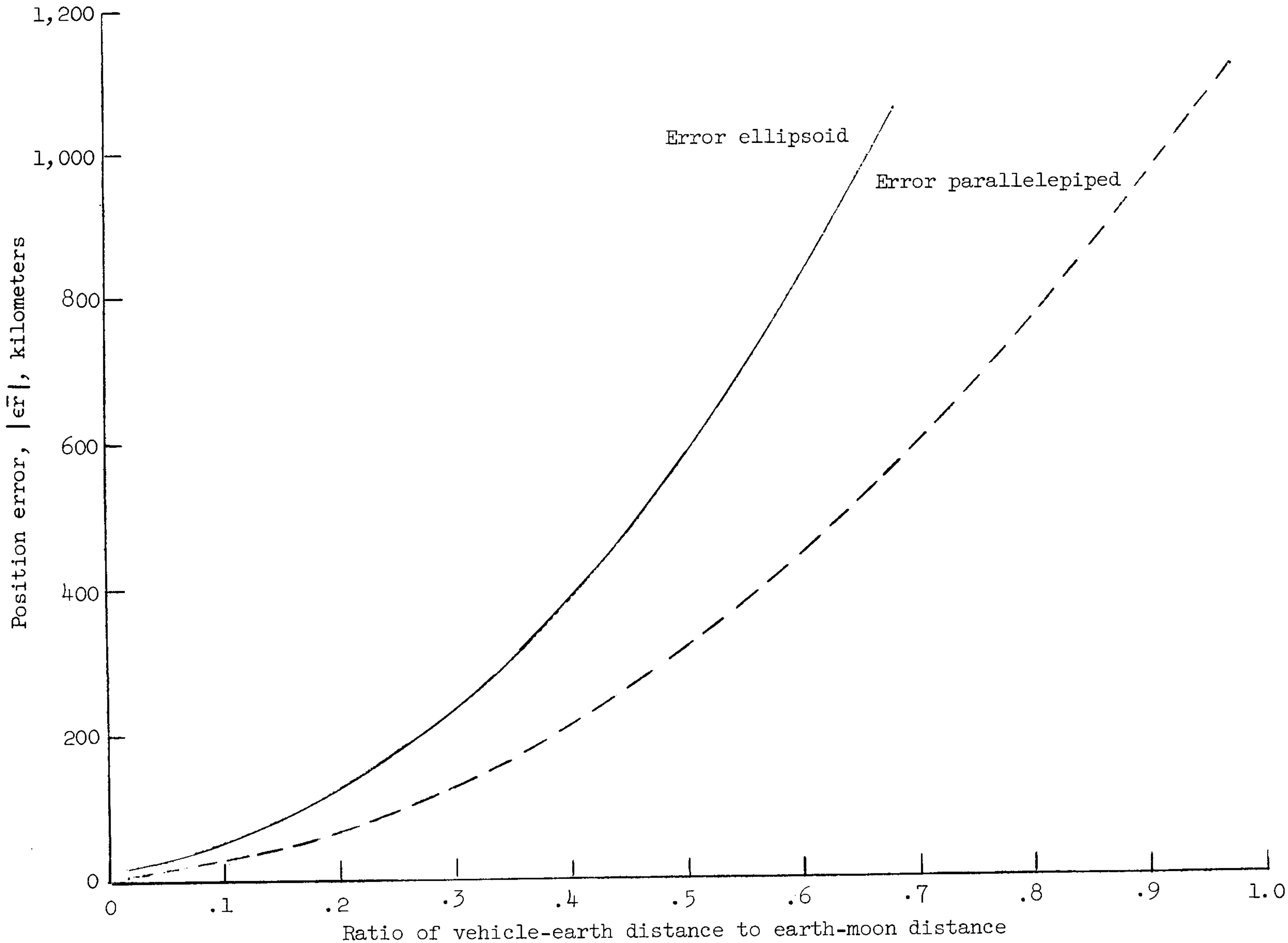


Figure 4.- Maximum position error $|\epsilon\bar{r}|$ calculated from measurements of angular diameter of earth and angles included at vehicle between each of two stars and earth's center.

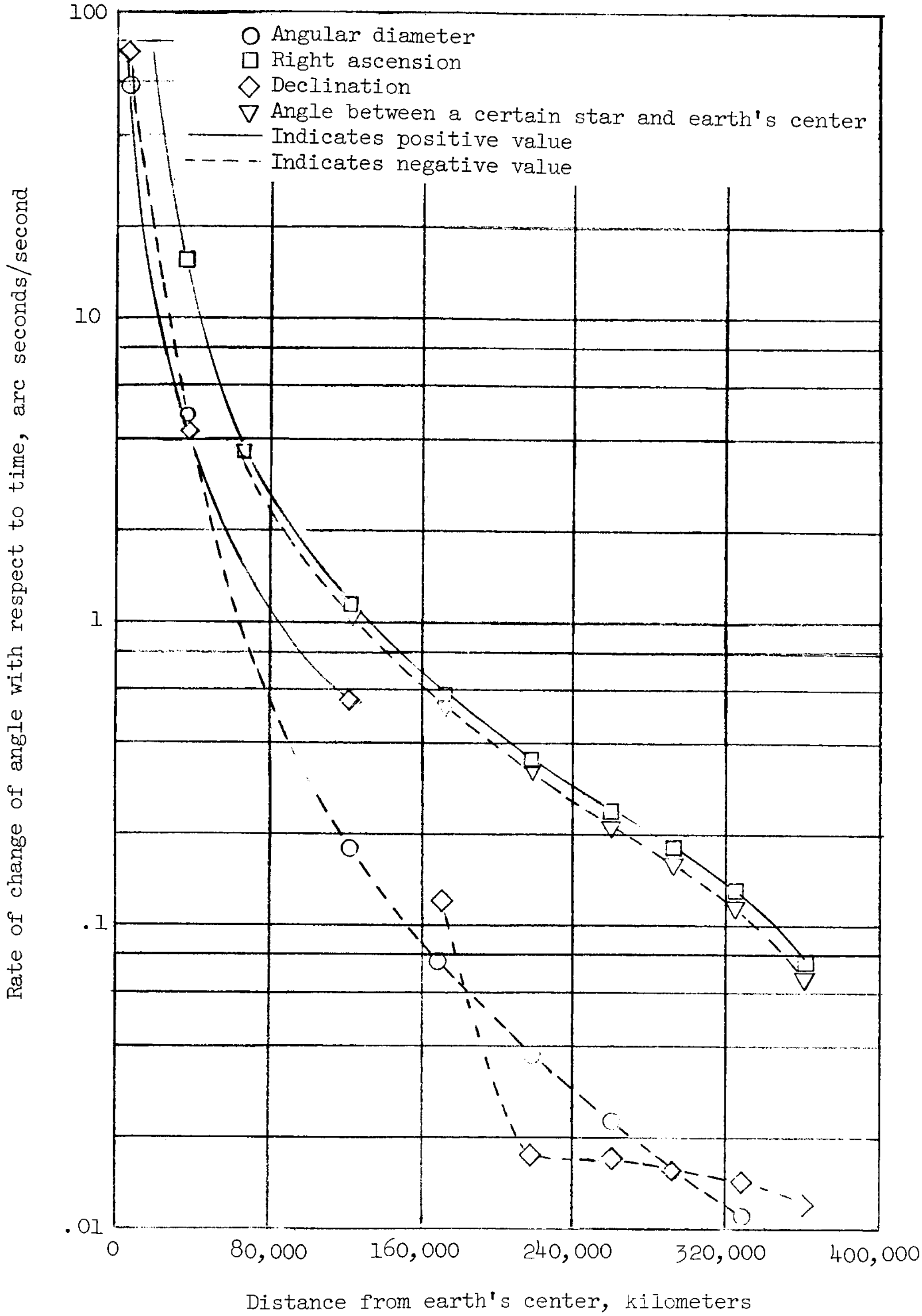


Figure 5.- Rate of change of various earth-referenced angular measurements with respect to time over a particular lunar trajectory.

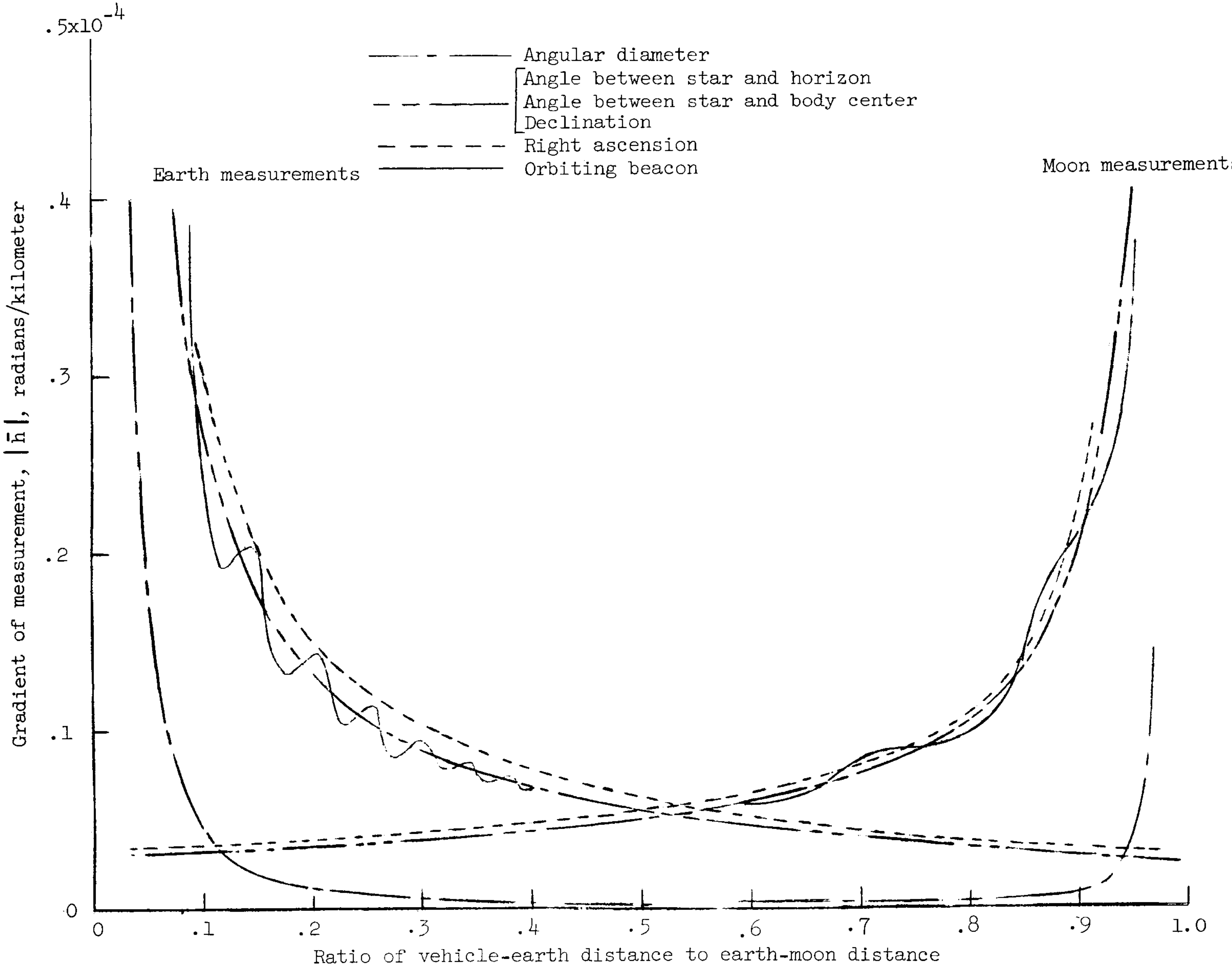


Figure 6.- Variation of gradients of various measurements with vehicle position on a particular lunar trajectory. Beacon in circular orbit having a radius of 8,047 kilometers.

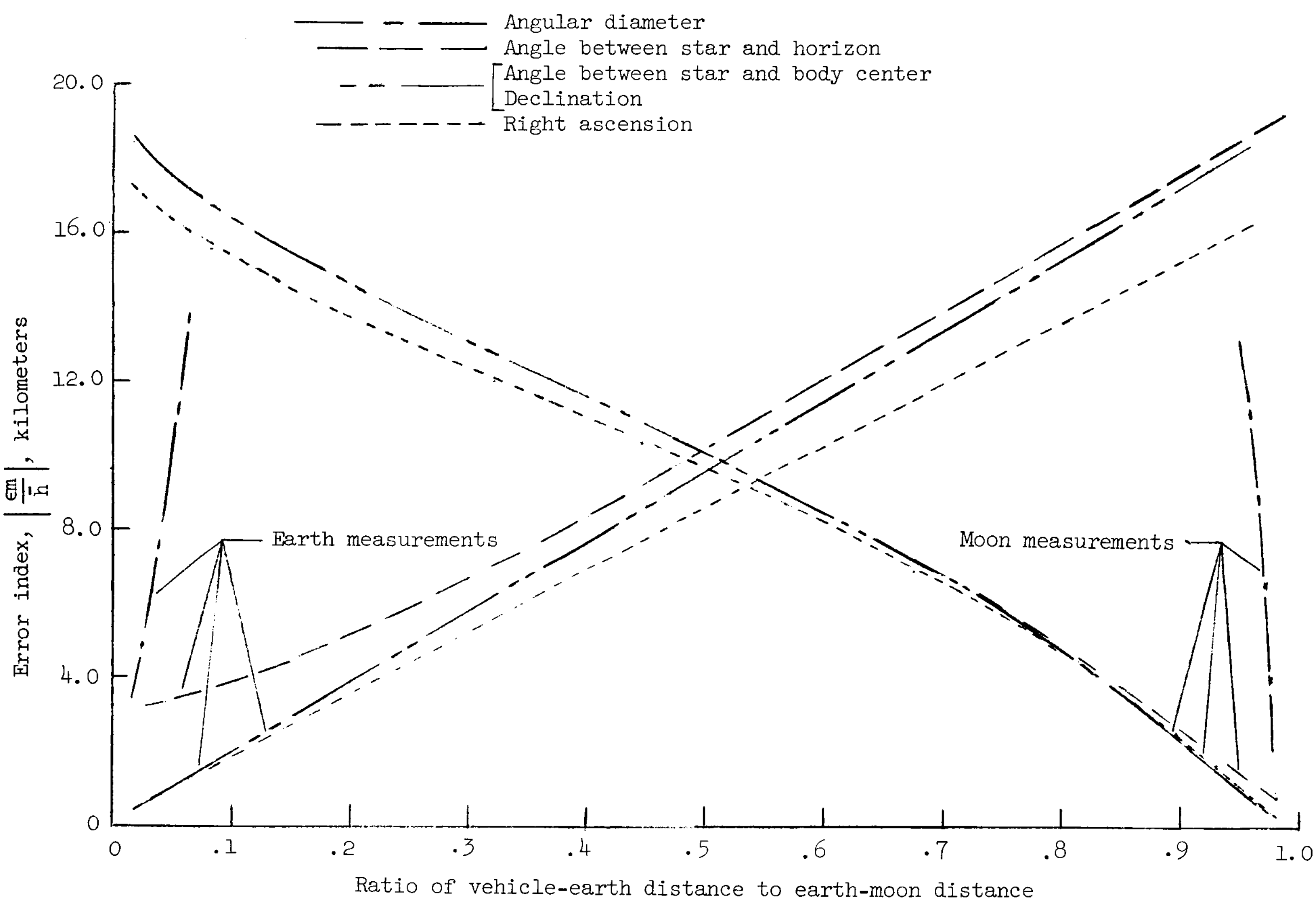


Figure 7.- Variation of error index $\left|\frac{\epsilon m}{\bar{h}}\right|$ with vehicle position on a particular lunar trajectory.

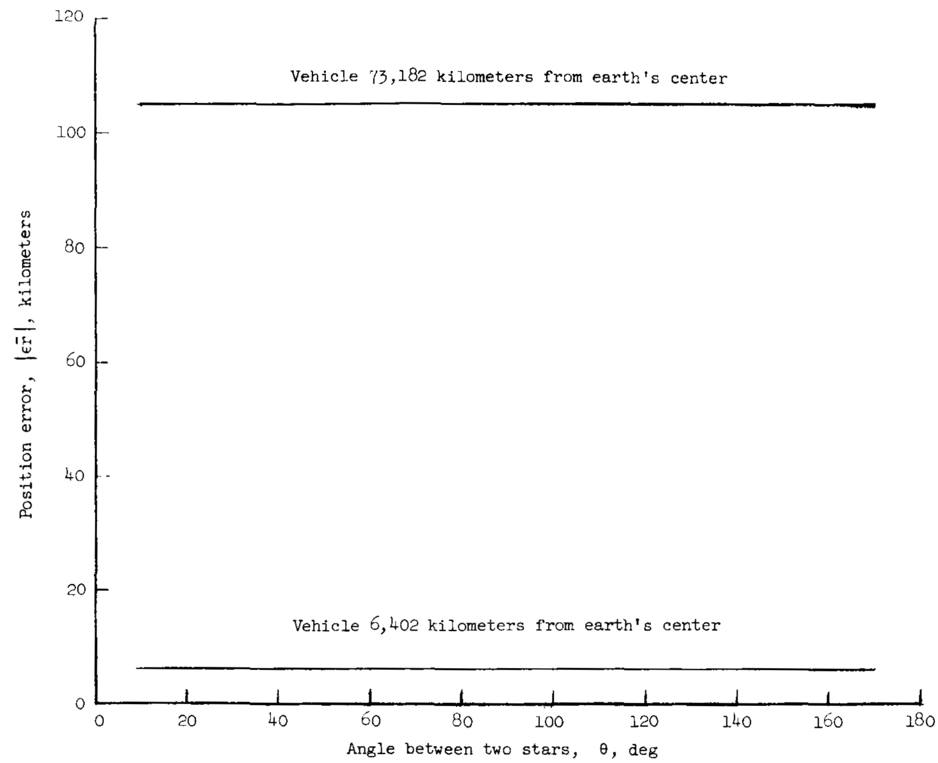


Figure 8.- Effect of angular separation θ of two stars on position error calculated from measurements of angular diameter and angles included at vehicle between each of two stars and earth's center.

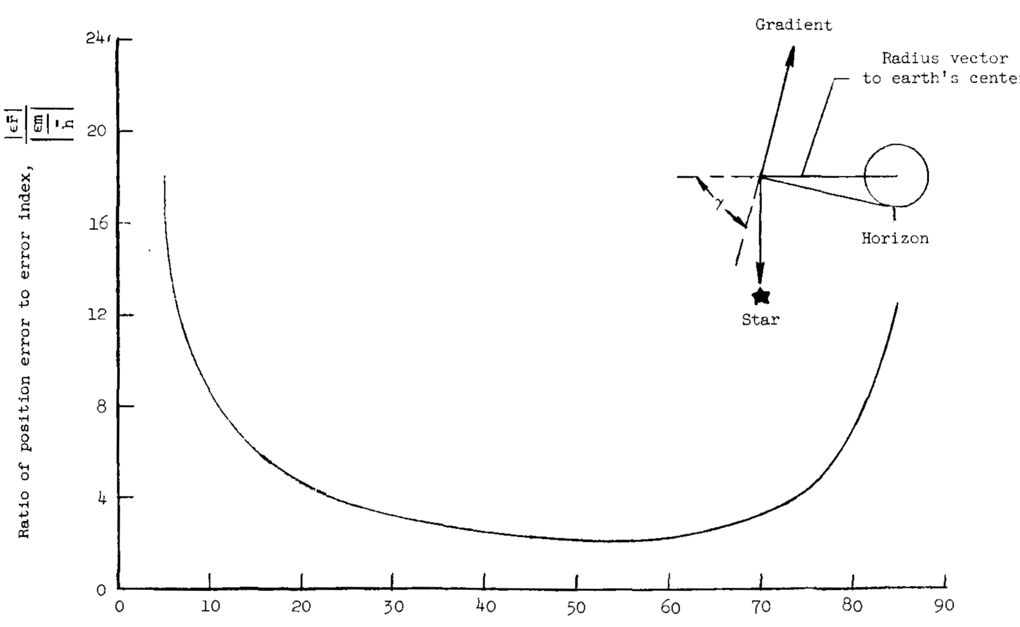


Figure 9.- Effect of γ on ratio of position error to error index.

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